EPFL - Automne 2020 — Y. Lodha, G. Buro

Smooth Manifolds — **EXAM**

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- L'examen est à livre ouvert, ce qui signifie que vous pouvez consulter tout les documents. Toutefois au cas où vous trouvez une référence qui répond à l'une des questions, vous devrez reformuler les calculs ou raisonnements avec vos mots et notations. Si votre réponse est très semblable à une référence externe vous êtes encouragé à citer cette source, il n'y aura pas de pénalité pour cela (par contre le plagiat n'est pas acceptable).
- Merci de rédiger vos réponses à la main, sur feuilles A4 recto uniquement (pas de recto-verso), en utilisant une plume un stylo ou un feutre assez foncé (noir, bleu).
- Pour chaque question principale utilisez une nouvelle feuille. Numérotez vos feuilles et indiquez votre nom sur chaque feuille. N'oubliez pas de rappeler le numéro de la question.
- Tous les raisonnements et calculs doivent être présentés dans votre rédaction.
- Lorsque vous avez terminé, faites des photos/scan de votre rédaction et convertissez en pdf (il existe des applications de scan pour smartphone (https://www.codeur.com/blog/application-scanner/). Vous devriez rassembler votre rédaction en un fichier pdf unique.
- Vous devez travailler de façon individuelle. Il est interdit de communiquer avec un autre étudiant ou toute autre personne pendant toutes la durée de l'examen.
- En remettant votre copie, vous vous engagez au respect du règlement EPFL sur les examens.
- L'enseignant pourra questionner les étudiants pour s'assurer qu'ils sont bien à l'origine du travail fourni(cela ne signifie pas forcément qu'il y a un soupçon de triche).
- Bon travail et bonne chance.

1. (10 points)

   (a) State the smooth manifold chart Lemma. Show that the topology that is given by this Lemma is Hausdorff and second countable.

   (b) Define the Grassman manifold $G_2(\mathbf{R}^3)$ and the smooth charts on it that satisfy the conditions of the smooth manifold chart Lemma. What is the dimension of this manifold?

   NOTE: You do not have to prove that the charts you describe for $G_2(\mathbf{R}^3)$ satisfy the smooth manifold chart Lemma, you simply have to define the charts.

2. (10 points) Describe $\mathbf{S}^{n-1}$ as a level set of a smooth map $\mathbf{R}^n \to \mathbf{R}$. Apply a theorem covered in the course to show that this provides a smooth structure on $\mathbf{S}^{n-1}$ that makes it a codimension 1 embedded submanifold in $\mathbf{R}^n$.

   NOTE: You need to check the hypothesis of the theorem you apply holds in this situation. You do not need to prove the theorem.

3. (15 points) Give examples of topological spaces with the following properties. In each case, prove that the properties hold.

   (a) A space X that is second countable, Hausdorff but not locally Euclidean.

   (b) A space X that is locally Euclidean, Hausdorff but not second countable.

   (c) A space X that is locally Euclidean, second countable but not Hausdorff.

4. (10 points) Let $\alpha, \beta, \gamma \in \Omega^1(\mathbb{R}^3)$ be given by

$$\alpha = xdx + ydy + zdz \qquad \beta = zdx + xdy + ydz \qquad \gamma = xydz$$

   and $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\phi(x, y) = (x + y, x^2, xy^2)$$

   Compute

   (a) $\alpha \wedge \beta$

   (b) $\alpha \wedge \gamma$

   (c) $\alpha \wedge \beta \wedge dx$

   (d) $d\alpha \wedge \gamma$

   (e) $\phi^*\alpha$

5. (15 points) Prove the following.

   (a) Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function. Show that $d^2 f = 0$. (Here d is the exterior derivative.)

   (b) Let $\omega = ycos(xy)dx + xcos(xy)dy$ be a 1-form on $\mathbf{R}^2$. Prove that ω is exact.

   (c) Let $\nu = xcos(xy)dx + ycos(xy)dy$ be a 1-form on $\mathbf{R}^2$. Prove that ν is not exact.

6. (10 points) Let M be an k-dimensional topological manifold. We define the set of *ordered* n-tuples of distinct points as

$$\mathrm{Conf}_n(M) = \{(x_1, ..., x_n) \mid x_1, ..., x_n \in M \text{ and } x_i \neq x_j \text{ whenever } i \neq j, 1 \leq i, j \leq n\}$$

   Show that $\mathrm{Conf}_n(M)$ is naturally a topological manifold. Compute the dimension of $\mathrm{Conf}_n(M)$.

7. (15 points) In this problem you will follow the steps below to prove the following theorem. The fixed point theorem states that given the ball

$$B^n = \{x = (x_1, ..., x_n) \in \mathbf{R}^n \mid ||x|| \leq 1\}$$

each smooth map $g : B^n \to B^n$ has a fixed point, i.e. there exists $x \in B^n$ such that $g(x) = x$.

(a) Let M be a compact n-dimensional, orientable, smooth manifold with boundary $\partial M \neq \emptyset$. Using partitions of unity, construct a differential $(n-1)$-form $\omega \in \Omega^{(n-1)}(\partial M)$ which is nowhere vanishing and satisfies that $d\omega = 0$.

(b) Apply the Stokes theorem using the differential form constructed in the previous part to show that there does not exist a smooth map

$$f : M \to \partial M$$

whose restriction to ∂M equals the identity.

(c) Show that every differentiable map $g : B^n \to B^n$ has a fixed point. Assume by way of contradiction that this is not the case. For each $x \in B^n$, consider the unique line that passes through $x, g(x)$. Use this to define a point $y_x \in \partial B^n$ and a map

$$B^n \to \partial B^n \qquad x \to y_x$$

that contradicts part 2.

Note: A proof by drawing a picture is sufficient. You do not need to write an explicit function.

8. (15 points) Let M be a connected differentiable n-manifold. For each $p \in M$, we denote by $\mathcal{O}_p$ as the set consisting of the two possible orientations on $T_p(M)$. Each element $O \in \mathcal{O}_p$ is called an *orientation at* p.

Define a manifold

$$\tilde{M} = \{(p, O) \mid p \in M, O \in \mathcal{O}_p\}$$

with the following charts. Given a chart (U, ϕ) for M, we define two charts (U_1, ϕ_1) and (U_2, ϕ_1) for $\tilde{M}$ as follows. Denote the two possible orientations on $U \subset M$ as O^1, O^2. Denote the restriction $O^i \restriction T_p(M)$ as O^i_p for each $p \in U$.

We define for each $1 \leq i \leq 2$

$$U_i = \{(p, O^i_p) \mid p \in U\}$$

and

$$\phi_i : U_i \to \mathbf{R}^n \qquad \phi_i(p, O^i_p) = \phi(p)$$

(a) Check that the sets

$$\{U_1, U_2 \mid (U, \phi) \text{ is a chart for } M\}$$

form a basis for a topology on $\tilde{M}$.

(b) Check that the map

$$\pi : \tilde{M} \to M \qquad \pi(p, O) = p$$

is a *generalised covering map*, i.e. for each $p \in M$, there is an open set U containing p such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\tilde{M}$, each of which is mapped homeomorphically to U via π.

(c) Show that $\tilde{M}$ is orientable.