Introduction to Differentiable Manifolds	
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Exercise Series 8 - Vector fields and flows	**2021–11–16**

Exercise 8.1. Compute the flows of the following vector fields.

(a) On the plane $\mathbb{R}^2$, the "angular" vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.
(b) A constant vector field X on the torus $\mathbb{T}^n$.

Exercise 8.2. Let X be a $\mathcal{C}^k$ tangent vector field on a manifold M, with $k \geq 1$.

(a) For a point $p \in M$ and numbers $s, t \in \mathbb{R}$, show that the equation $\Phi_X^{(s+t)}(p) = \Phi_X^t(\Phi_X^s(p))$ holds if the right-hand side is defined.
(b) We say that X is **complete** if $\Phi_X^t(p)$ is defined for all $t \in \mathbb{R}$. Show that X is complete if its support is a compact set. In particular, on a compact manifold, every vector field is complete.
(c) If X is complete, show that the map Φ_X^t is a diffeomorphism $M \to M$.

Exercise 8.3. If X is a complete $\mathcal{C}^k$ vector field with $(k \geq 1)$ and $h \in \mathcal{C}^{k+1}(M, \mathbb{R})$.

(a) Show that the function $X(h) : M \to \mathbb{R}$ that sends $p \mapsto X_p(h)$ is $\mathcal{C}^k$.
(b) Show that $X(h) = \frac{\partial}{\partial t}\big|_{t=0} h_t$, where $h_t = (\Phi_X^t)^*(h) = h \circ \Phi_X^t$.
Also show that $X(h_t) = (\Phi_X^t)^*(X(h))$.

Exercise 8.4. Let $f : M \to N$ be a smooth map. A vector field $X \in \mathfrak{X}(M)$ is f**-related** to a vector field $Y \in \mathfrak{X}(N)$ if $\mathrm{T}_p f(X_p) = Y_{f(p)}$ for all $p \in M$.

(a) X is f-related to Y if and only if $X_p(f \circ h) = Y_{f(p)}(h)$ for all functions $h \in \mathcal{C}^\infty(N, \mathbb{R})$ and all points $p \in M$.
(b) If X is f-related to Y and γ is an integral curve of X, show that $f \circ \gamma$ is an integral curve of Y.
(c) If f is a local diffeo, for every vector field $Y \in \mathfrak{X}(N)$ there exists a unique $X \in \mathfrak{X}(M)$ that is f-related to Y. We denote $f^*Y := X$.
Thus if f is a diffeo, f-relatedness is a bijection from $\mathfrak{X}(M)$ to $\mathfrak{X}(N)$. In this case, if X is f-related to Y, we write $X = f^*Y$ and $Y = f_*X$.
(d) If f is a closed embedding, show that every vector field $X \in \mathfrak{X}(M)$ is f-related to some vector field $Y \in \mathfrak{X}(N)$.
Hint: Construct Y locally, then use partitions of unity.
What happens if f is just an immersion? In this case, find and prove a local version of the fact.
(e) A vector field $X \in \mathfrak{X}(M)$ is **tangent** to a smooth submanifold $S \subseteq M$ if $X_p \in T_pS$ for all points $p \in S$. If this happens and in addition S is closed, show that every integral curve of X that visits S is contained in S.

Exercise 8.5. If X is a smooth vector field on a manifold M and $p \in M$ is a point where $X_p \neq 0$, then there exists a chart (U, ϕ) of M defined at p such $X|_U = \frac{\partial}{\partial \phi^0}$. ***Hint:*** It is easier to construct the inverse $\psi = \phi^{-1}$. Use a function of the form $\psi(x) = \Phi_X^{x^0}(f(x^1, \ldots, x^{n-1}))$, where $f : U \to M$ is a suitable function defined on an open set $U \subseteq \mathbb{R}^{n-1}$.