Introduction to Differentiable Manifolds

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Exercise series 5 | **2021–10–19**

Exercise 5.1. Consider the map

$$f : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (2 + \tanh t) \cdot (\cos t, \sin t).$$

Show that f is an injective immersion. Is it a smooth embedding?

Exercise 5.2. Consider the following subsets of $\mathbb{R}^2$. Which is an embedded submanifold ? Which is the image of an immersion ?

(a) The "cross" $S := \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$.
(b) The "corner" $C := \{(x, y) \in \mathbb{R}^2 \mid xy = 0, x \geq 0, y \geq 0\}$

Exercise 5.3. Let N be a $\mathcal{C}^k$-embedded n-submanifold of some m-manifold M, with $k \geq 1$. Show that there exists an open set $U \subseteq M$ that contains N as a closed subset.

Exercise 5.4. Let $f : M \to N$ be an injective immersion of $\mathcal{C}^k$ manifolds. Show that there exists a closed embedding $M \to N \times \mathbb{R}$.
Hint: Recall that there exists a proper map $g : M \to \mathbb{R}$.

Exercise 5.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^3 + y^3 + 1$.

(a) What are the regular values of f? For which $c \in \mathbb{R}$ is the level set $f^{-1}(\{c\})$ an embedded submanifold of $\mathbb{R}^2$?
(b) In the case where $S = f^{-1}(\{c\})$ is an embedded submanifold, $p \in S$, write down an equation for the tangent space $\iota_*(T_pS) \subset T_p\mathbb{R}^2$ where as usual we identify $T_p\mathbb{R}^2 \cong \mathbb{R}^2$ (i.e. you are expected to write down the equation for a line in $\mathbb{R}^2$).

Exercise 5.6. Consider the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the projection map.

(a) Give $\mathbb{T}^n$ a natural smooth structure so that π is a local diffeomorphism.
(b) Show that a map $f : \mathbb{T}^n \to N$ (where M is a smooth manifold) is $\mathcal{C}^k$ if and only if the composite $f \circ \pi$ is smooth.
(c) Show that $\mathbb{T}^n$ is diffeomorphic to the product of n copies of the circle $\mathbb{S}^1$.

Exercise 5.7. Show that the map $g : \mathbb{T}^2 \to \mathbb{R}^3$ given by

$$g([s, t]) = ((2 + \cos s) \cos t, (2 + \cos s) \sin t, \sin s)$$

is a smooth embedding of the 2-torus in $\mathbb{R}^3$.

Exercise 5.8. Show that the following subgroups of $GL_n(\mathbb{R})$ are closed submanifolds. Compute their dimension and their tangent space at the identity.

(a) The *special linear group* $\mathrm{SL}_n(\mathbb{R})$, consisting of matrices with determinant equal to 1.
(b) The *orthogonal group* $O_n(\mathbb{R})$, consiting of orthogonal matrices A (which satisfy $A^tA = I$).
Hint: Consider the map $f : M(n) \to M_{sym}(n)$ that sends $A \mapsto A^tA$, there $M_{sym}(n)$ is the vector space of *symmetric* $n \times n$ matrices.