Introduction to Differentiable Manifolds

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Exercise Series 13 - Exterior derivative, Stokes' theorem 2021–12–21

Exercise 13.1. Prove Proposition 7.2.6. (properties of the integral: linearity, etc).

Exercise 13.2. Prove that a continuous k-form is determined by the value of its integrals (Proposition 7.3.12). ***Hint:*** Use a chart to move the problem to $\mathbb{R}^n$, then integrate on small pieces of coordinate planes.

Exercise 13.3.* Let $f : M \to N$ be a smooth map between smooth manifolds. Then for all $\omega \in \Omega^k(M)$ we have

$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega).$$

Exercise 13.4.* Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$ and let (v, w) be the standard coordinates on $\mathbb{R}^2$. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as $\phi(x, y, z) = (x + z, xy)$. Let $\alpha = e^w \,\mathrm{d}v + v \,\mathrm{d}w$ and $\beta = v \ \mathrm{d}v \wedge \mathrm{d}w$ be 2-forms on $\mathbb{R}^2$. Compute the following differential forms:

$$\alpha \wedge \beta, \quad \phi^*(\alpha), \quad \phi^*(\beta), \quad \phi^*(\alpha) \wedge \phi^*(\beta).$$

Exercise 13.5.* Compute the exterior derivative of the following forms:

(a) on $\mathbb{R}^2 \setminus \{0\}$ $\theta = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$.
(b) on $\mathbb{R}^3$, $\varphi = \cos(x)\,\mathrm{d}y \wedge \mathrm{d}z$.
(c) on $\mathbb{R}^3$ $\omega = A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z$.

Exercise 13.6.* Deduce the following classical theorems from Stokes' theorem.

(a) **Green's theorem.** Let $D \subseteq \mathbb{R}^2$ be a smooth 2-dimensional compact embedded submanifold with boundary in $\mathbb{R}^2$. Then for any differentiable 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ defined on an open neighborhood of D we have

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \ \mathrm{d}y = \int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

(b) **Divergence theorem.** Let $A \subset \mathbb{R}^3$ be a 3-dimensional compact embedded submanifold with boundary in $\mathbb{R}^3$. Then for any smooth vector field $F : A \to \mathbb{R}^3$ we have

$$\int_A \mathrm{div}F \ \mathrm{d}V = \int_{\partial A} F \cdot \mathrm{d}$$

wher $\mathrm{d}V := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the standard 3-form on $\mathbb{R}^3$ and on the right hand side we have the formal inner product with $\mathrm{d}S = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$.