Exercise 10.1. On the plane $\mathbb{R}^2$ with the standard coordinates (x, y) consider the 1-form $\theta = x\,\mathrm{d}y$. Compute the integral of θ along each side of the square $[1, 2] \times [3, 4]$, with each of the two orientations.

Exercise 10.2. Let $\mathcal{B} = (E_i)_i$ and $\widetilde{\mathcal{B}} = (\widetilde{E}_j)_j$ be two bases of a vector space $V \simeq \mathbb{R}^n$, and let $\mathcal{B}^* = (\varepsilon^i)_i$ and $\widetilde{\mathcal{B}}^* = (\widetilde{\varepsilon}^j)_j$ be the respective dual bases. Note that a tensor $T \in \mathrm{Ten}^k V$ can be written as

$$T = \sum_{i_0,\dots,i_{k-1}} T_{i_0,\dots,i_{k-1}}\, \varepsilon^{i_0} \otimes \varepsilon^{i_{k-1}} \quad \text{or as} \quad T = \sum_{j_0,\dots,j_{k-1}} \widetilde{T}_{j_0,\dots,j_{k-1}}\, \widetilde{\varepsilon}^{j_0} \otimes \widetilde{\varepsilon}^{j_{k-1}}.$$

Find the transformation law that expresses the coefficients $\widetilde{T}_{j_0,\dots,j_{k-1}}$ in terms of the coefficients $T_{i_0,\dots,i_{k-1}}$.

Exercise 10.3 (Alternating covariant tensors)**.** Let V be a finite-dimensional real vector space.

(a) Let $T \in \mathrm{Ten}^k V$. Suppose that with respect to some basis ε^i of V^*

$$T = \sum_{1 \le i_1,\dots,i_k < n} T_{i_1\cdots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}.$$

Show that T is alternating iff for all $\sigma \in S_k$: $T_{i_{\sigma(1)}\cdots i_{\sigma(k)}} = \mathrm{sgn}(\sigma)\, T_{i_1\cdots i_k}$.

(b) Show that for any covectors $\omega^1, \dots, \omega^k \in V^*$ and vectors $X_1, \dots, X_k \in V$ we have

$$\omega^1 \wedge \cdots \wedge \omega^k(X_1, \dots, X_k) = \det(\omega^i(X_j)).$$

Exercise 10.4 (Some practice with the wedge product)**.** Let V be a finite-dimensional vector space over $\mathbb{R}$.

(i) Show that the covectors $\omega^1, \dots, \omega^k \in V^*$ are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

(ii) Let $\{\omega^1, \dots, \omega^k\}$ and $\{\eta^1, \dots, \eta^k\}$ both be sets of k independent covectors. Show that they span the same subspace if and only if

$$\omega^1 \wedge \cdots \wedge \omega^k = c\,\eta^1 \wedge \cdots \wedge \eta^k$$

for some nonzero real number c.

(iii) On the space $V = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, let $(\alpha^0, \dots, \alpha^{n-1},\ \beta^0, \dots, \beta^{n-1})$ be the dual of the standard base. Consider the alternating 2-tensor

$$\omega = \sum_i \alpha^i \wedge \beta^i \in \mathrm{Alt}^2 V.$$

Compute the $2n$-tensor

$$\frac{1}{n!} \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}}.$$